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Integrable systems in ellipsoidal coordinates

M Gadella[1], J Negro[1] and G P Pronko[2,3]

[1] Departamento de Física Teórica, Facultad de Ciencias, 47011 Valladolid, Spain
[2] Institute for High Energy Physics, Protvino Moscow Region, Russia
[3] Institute of Nuclear Physics, National Research Center 'Demokritos', Athens, Greece

E-mail: manuelgadella@yahoo.com.ar, jnegro@fta.uva.es and pronko@ihep.ru

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Abstract
The most general potential for complete integrability of three-dimensional classical and quantum problems is obtained starting from a certain class of two second-order integrals of motion. We also show that these potentials lead to separation of variables in both classical and quantum cases.

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1. Introduction

In a previous article, we have studied three-dimensional integrable systems with axial symmetry [1]. In the present paper, we consider a class of completely integrable systems in the three-dimensional Euclidean space by using ellipsoidal coordinates. The proposed procedure will be valid in the contexts of classical and quantum mechanics as well. In both cases, one reaches complete integrability and then gets separation of variables. In this sense, we have an approach opposite to the general method developed by Eisenhart [2] who started from the problem of separation of variables. Here, we must quote the pioneering systematic work by Smorodinsky and collaborators [3] on the search for systems allowing for separation of variables using the group approach (see also [4, 5]).

The problem we have solved here is the following: complete integrability of a three-dimensional classical problem with the Hamiltonian function $H = \mathbf{p}^2/2m + U(\mathbf{x})$ requires finding two new constants of motion. If we assume that these constants of motion have the form $H_1 = p_i g_1^{ik}(\mathbf{x}) p_k/2m + U_1(\mathbf{x})$ and $H_2 = p_i g_2^{ik}(\mathbf{x}) p_k/2m + U_2(\mathbf{x})$, i.e., they are of Hamiltonian form for which the kinetic part is quadratic in momenta, we show that there are always a class of solutions in ellipsoidal coordinates.

This classical situation has an immediate translation in the quantum case, in which we show that if the potentials are taken as in the classical case, separation of variables follows. The final result shows that the three-dimensional system can be reduced to three uncoupled similar Schrödinger-type equations.

The classical and quantum approaches presented here are simple, self-contained and give a complete answer to the proposed problem.

This paper is organized as follows: after the introduction of ellipsoidal coordinates in section 2, we present our strategy of search for complete integrability for the classical case in section 3. Thus, we find two second-order integrals of motion in involution with a conventional Hamiltonian of the form $\mathbf{p}^2/2m + U(\mathbf{x})$ and among themselves. Section 4 is devoted to the extensions of the previous results to the quantum case.

2. Ellipsoidal coordinates

Let us assume that a, b and c are three fixed positive numbers and λ, μ and ν are real numbers such that the following relation holds among them:

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2. \tag{1}$$

This relation determines whether the following equation represents either an ellipsoid:

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \tag{2}$$

or a one sheeted hyperboloid:

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1, \tag{3}$$

or a two sheeted hyperboloid:

$$\frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} + \frac{z^2}{c^2+\nu} = 1. \tag{4}$$

Note that after relations (1), the sign of the coefficient of z^2 in (3) is minus and the signs of the coefficients of y^2 and z^2 in (4) are also minus. Solving (2)–(4) in x^2, y^2 and z^2, we obtain:

$$x^2 = \frac{(\lambda+a^2)(\mu+a^2)(\nu+a^2)}{(b^2-a^2)(c^2-a^2)}, \tag{5}$$

$$y^2 = \frac{(\lambda+b^2)(\mu+b^2)(\nu+b^2)}{(c^2-b^2)(a^2-b^2)} \tag{6}$$

and

$$z^2 = \frac{(\lambda+c^2)(\mu+c^2)(\nu+c^2)}{(a^2-c^2)(b^2-c^2)}. \tag{7}$$

Equations (5)–(7) provide a system of ellipsoidal coordinates that are λ, μ and ν. Inversion of these formulae gives ellipsoidal coordinates in terms of Euclidean coordinates. We can also obtain the following relations:

$$A = \mu + \nu + \lambda = x^2 + y^2 + z^2 - (a^2 + b^2 + c^2), \tag{8}$$

$$B = \lambda\mu + \lambda\nu + \mu\nu = (a^2b^2 + a^2c^2 + b^2c^2) - x^2(b^2 + c^2) - y^2(a^2 + c^2) - z^2(a^2 + b^2), \tag{9}$$

$$C = \mu\nu\lambda = x^2b^2c^2 + y^2a^2c^2 + z^2a^2b^2 - a^2b^2c^2. \tag{10}$$

These relations show that the numbers λ, μ and ν are the roots of the following equation:

$$z^3 - Az^2 + Bz - C = 0. \tag{11}$$

3. Searching for integrability

Next, let us assume that we have a classical system for which the motion is governed by the following Hamiltonian:

$$H := \frac{1}{2m}\mathbf{p}^2 + U(\mathbf{x}) = \frac{1}{2m} p_i g^{ij}(\mathbf{x}) p_j + U(\mathbf{x}), \qquad g^{ij}(\mathbf{x}) = \delta^{ij}, \tag{12}$$

$i = 1, 2, 3$, where δ^{ij} is the Kronecker delta. If the system is integrable, two additional constants of motion here given by H_1 and H_2 must exist. These constants of motion should satisfy the following relations:

$$\{H, H_1\} = \{H, H_2\} = \{H_1, H_2\} = 0, \tag{13}$$

where $\{\cdot,\cdot\}$ in (13) denotes Poisson brackets. We shall look for systems such that the remaining integrals of motion are quadratic in momentum. Thus, a typical ansatz for H_1 and H_2 could be

$$H_1 = \frac{1}{2m} p_i g_1^{ik}(\mathbf{x}) p_k + U_1(\mathbf{x}) \tag{14}$$

and

$$H_2 = \frac{1}{2m} p_i g_2^{ik}(\mathbf{x}) p_k + U_2(\mathbf{x}), \tag{15}$$

where repeated indices indicate sum from 1 to 3. The 'metrics' g_1^{ik} and g_2^{ik} as well as the 'potential terms' $\{U, U_1, U_2\}$ must be found from relations (13). As becomes obvious from (13), the quadratic terms in H, H_1 and H_2 should also be in involution. Since potential terms are in involution by definition, one has to impose the following conditions to provide (13):

$$0 = \{H, H_1\} = \frac{1}{2m}\left(p_i\partial^i U_1 + \partial^i U_1 p_i - p_i g_1^{ik}\partial_k U - g^{ik}\partial_k U p_i\right) \tag{16}$$

$$0 = \{H, H_2\} = \frac{1}{2m}\left(p_i\partial^i U_2 + \partial^i U_2 p_i - p_i g_2^{ik}\partial_k U - g_2^{ik}\partial_k U p_i\right) \tag{17}$$

$$0 = \{H_1, H_2\} = \frac{1}{2m}\left(p_i g^{ik}\partial_k U_2 + g_1^{ik}\partial_k U_1 p_i - p_i g_2^{ik}\partial_k U_1 - g_2^{ik}\partial_k U_1 p_i\right), \tag{18}$$

where again we sum over repeated indices and use the notation

$$\partial_i = \partial^i = \frac{\partial}{\partial x_i}. \tag{19}$$

The display of repeated terms in (16)–(18) would facilitate the comparison with the quantum case where the order is important. This notation makes sense as the x_i are Cartesian coordinates. Equations (16)–(18) immediately yield

$$\partial^i U_1 = g_1^{ik}\partial_k U \tag{20}$$

$$\partial^i U_2 = g_2^{ik}\partial_k U \tag{21}$$

$$g_1^{ik}\partial_k U_2 = g_2^{ik}\partial_k U_1. \tag{22}$$

The metrics $g_1^{ik}(\mathbf{x})$ and $g_2^{ik}(\mathbf{x})$ will be fixed using the property of involution of the 'kinetic terms' of H, H_1 and H_2. For $g_1^{ik}(\mathbf{x})$, this condition gives

$$\left\{p_i g_1^{ik}(\mathbf{x}) p_k, \mathbf{p}^2\right\} = 0, \tag{23}$$

where the bracket denotes the Poisson bracket. This condition implies that $p_i g_1^{ik}(\mathbf{x}) p_k$ should be a quadratic function of the coordinates and therefore, it can be expressed as a linear combination of the squares of components L_i of the angular momentum and the momentum, p_i. We choose this linear combination to be in the following form:

$$p_i g_1^{ik}(\mathbf{x}) p_k := \mathbf{L}^2 - p_1^2(b^2+c^2) - p_2^2(a^2+c^2) - p_3^2(a^2+b^2), \tag{24}$$

so that if $G_1(\mathbf{x})$ is the matrix whose matrix elements are given by $g_1^{ik}(\mathbf{x})$, we have

$$G_1(\mathbf{x}) = \begin{pmatrix} z^2+y^2-(a^2+c^2) & -xy & -xz \\ -xy & x^2+z^2-(a^2+c^2) & -xy \\ -xz & -yz & x^2+y^2-(a^2+b^2) \end{pmatrix}. \tag{25}$$

Note that matrix (25) is symmetric and therefore diagonalizable. Then, it admits three different eigenvectors. The eigenvectors and their corresponding eigenvalues are given by

$$E_\lambda := (\partial_x\lambda, \partial_y\lambda, \partial_z\lambda); \quad \mu+\nu \tag{26}$$

$$E_\mu := (\partial_x\mu, \partial_y\mu, \partial_z\mu); \quad \lambda+\nu \tag{27}$$

$$E_\nu := (\partial_x\nu, \partial_y\nu, \partial_z\nu); \quad \lambda+\mu. \tag{28}$$

Note that ellipsoidal coordinates depend on (x, y, z) through (8)–(10), so that the partial derivatives in (26)–(28) make sense. The eigenvectors can be written in explicit form as

$$\partial_x\lambda = \frac{1}{\Delta}\frac{2}{x}\left[\frac{1}{\mu+b^2}\frac{1}{\nu+c^2} - \frac{1}{\mu+c^2}\frac{1}{\nu+b^2}\right] \tag{29}$$

$$\partial_y\lambda = \frac{1}{\Delta}\frac{2}{y}\left[\frac{1}{\mu+c^2}\frac{1}{\nu+a^2} - \frac{1}{\mu+a^2}\frac{1}{\nu+c^2}\right] \tag{30}$$

$$\partial_z\lambda = \frac{1}{\Delta}\frac{2}{z}\left[\frac{1}{\mu+a^2}\frac{1}{\nu+b^2} - \frac{1}{\mu+b^2}\frac{1}{\nu+a^2}\right] \tag{31}$$

$$\partial_x\mu = \frac{1}{\Delta}\frac{2}{x}\left[\frac{1}{\nu+b^2}\frac{1}{\lambda+c^2} - \frac{1}{\nu+c^2}\frac{1}{\lambda+b^2}\right] \tag{32}$$

$$\partial_y\mu = \frac{1}{\Delta}\frac{2}{y}\left[\frac{1}{\nu+c^2}\frac{1}{\lambda+a^2} - \frac{1}{\nu+a^2}\frac{1}{\lambda+c^2}\right] \tag{33}$$

$$\partial_z\mu = \frac{1}{\Delta}\frac{2}{z}\left[\frac{1}{\nu+a^2}\frac{1}{\lambda+b^2} - \frac{1}{\nu+b^2}\frac{1}{\lambda+a^2}\right] \tag{34}$$

$$\partial_x\nu = \frac{1}{\Delta}\frac{2}{x}\left[\frac{1}{\lambda+b^2}\frac{1}{\mu+c^2} - \frac{1}{\lambda+c^2}\frac{1}{\mu+b^2}\right] \tag{35}$$

$$\partial_y\nu = \frac{1}{\Delta}\frac{2}{y}\left[\frac{1}{\lambda+c^2}\frac{1}{\mu+a^2} - \frac{1}{\lambda+a^2}\frac{1}{\mu+c^2}\right] \tag{36}$$

$$\partial_z\nu = \frac{1}{\Delta}\frac{2}{z}\left[\frac{1}{\lambda+a^2}\frac{1}{\mu+b^2} - \frac{1}{\lambda+b^2}\frac{1}{\mu+a^2}\right], \tag{37}$$

where

$$\Delta = \det\begin{vmatrix} \frac{1}{\lambda+a^2} & \frac{1}{\mu+a^2} & \frac{1}{\nu+a^2} \\ \frac{1}{\lambda+b^2} & \frac{1}{\mu+b^2} & \frac{1}{\nu+b^2} \\ \frac{1}{\lambda+c^2} & \frac{1}{\mu+c^2} & \frac{1}{\nu+c^2} \end{vmatrix}. \tag{38}$$

We can easily check that E_λ, E_μ and E_ν are mutually orthogonal, so that

$$E_\lambda \cdot E_\mu = E_\lambda \cdot E_\nu = E_\mu \cdot E_\nu = 0. \tag{39}$$

We recall that these eigenvectors of $G_1(\mathbf{x})$ depend on the coordinates $\mathbf{x} = (x, y, z)$.

Our next goal is to obtain the matrix elements $g_2^{ik}(\mathbf{x})$. First, note that the involution conditions

$$\left\{p_i g_2^{ik}(\mathbf{x}) p_k, \, p_i g_1^{ik}(\mathbf{x}) p_k\right\} = \left\{p_i g_2^{ik}(\mathbf{x}) p_k, \mathbf{p}^2\right\} = 0, \tag{40}$$

mean that $p_i g_2^{ik}(\mathbf{x}) p_k$ have to be a linear combination of ${L_i}^2$ and ${p_i}^2$, $i = 1, 2, 3$. This linear combination can be chosen as

$$p_i g_2^{ik}(\mathbf{x}) p_k = -\left(L_1^2 a^2 + L_2^2 b^2 + L_3^2 c^2\right) + p_1^2 b^2 c^2 + p_2^2 a^2 c^2 + p_3^2 a^2 b^2, \tag{41}$$

leading to

$$g_2^{ik} = \left(g_1^2\right)^{ik} - (\lambda + \mu + \nu) g_1^{ik} + (\lambda\mu + \lambda\nu + \mu\nu)\delta^{ik}, \tag{42}$$

where $\left(g_1^2\right)^{ik}$ and δ^{ik} are the ik component of the squared matrix $G_1^2(\mathbf{x})$ and the Kronecker delta, respectively. This allows us to write the matrix $G_2(\mathbf{x})$, for which the matrix elements are $g_2^{ik}(\mathbf{x})$, in the following form:

$$G_2(\mathbf{x}) = \begin{pmatrix} -c^2 y^2 - b^2 z^2 + b^2 c^2 & c^2 xy & b^2 xz \\ c^2 xy & -c^2 x^2 - a^2 z^2 + a^2 c^2 & a^2 yz \\ b^2 xz & a^2 yz & -b^2 x^2 - a^2 y^2 + a^2 b^2 \end{pmatrix}. \tag{43}$$

The eigenvectors and their corresponding eigenvalues of $G_2(\mathbf{x})$ are given by

$$E_\lambda; \quad \mu\nu \tag{44}$$

$$E_\mu; \quad \nu\lambda \tag{45}$$

$$E_\nu; \quad \lambda\mu. \tag{46}$$

The vectors E_λ, E_μ and E_ν have respective norms given by

$$\|E_\lambda\|^2 = \frac{4(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)}{(\lambda - \mu)(\lambda - \mu)} \tag{47}$$

$$\|E_\mu\|^2 = \frac{4(\mu + a^2)(\mu + b^2)(\mu + c^2)}{(\mu - \nu)(\mu - \lambda)} \tag{48}$$

$$\|E_\nu\|^2 = \frac{4(\nu + a^2)(\nu + b^2)(\nu + c^2)}{(\nu - \mu)(\nu - \lambda)}. \tag{49}$$

After having fixed the 'kinetic terms' of H_1 and H_2, our next step is to obtain the 'potential terms' $U(\mathbf{x})$, $U_1(\mathbf{x})$ and $U_2(\mathbf{x})$. This can be obtained from equations (20)–(22). For this purpose, we shall work in ellipsoidal coordinates λ, μ, ν. The relation between the derivatives with respect to the Cartesian coordinates and the derivatives with respect to the ellipsoidal coordinates can be expressed as

$$\partial^i \equiv \partial_i = (\partial_i\lambda)\partial_\lambda + (\partial_i\mu)\partial_\mu + (\partial_i\nu)\partial_\nu. \tag{50}$$

Then, equation (20) takes the following form in terms of λ, μ, ν:

$$(\partial_i\lambda)\partial_\lambda U_1 + (\partial_i\mu)\partial_\mu U_1 + (\partial_i\nu)\partial_\nu U_1 = g_1^{ik}(\partial_k\lambda)\partial_\lambda U + g_1^{ik}(\partial_k\mu)\partial_\mu U + g_1^{ik}(\partial_k\nu)\partial_\nu U \tag{51}$$

(where we have omitted the dependence on $\mathbf{x}$ in order to alleviate the notation) and then using (26)–(28) and the short notation $U_\lambda = \partial_\lambda U$, $U_\mu = \partial_\mu U$, $U_\nu = \partial_\nu U$, we have that

$$\begin{aligned}(U_1)_\lambda E_\lambda + (U_1)_\mu E_\mu + (U_1)_\nu E_\nu &= U_\lambda G_1(\mathbf{x}) E_\lambda + U_\mu G_1(\mathbf{x}) E_\mu + U_\nu G_1(\mathbf{x}) E_\nu \\ &= (\mu+\nu) U_\lambda E_\lambda + (\nu+\lambda) U_\mu E_\mu + (\mu+\lambda) U_\nu E_\nu.\end{aligned} \tag{52}$$

The same procedure in (21) and (22) yields respectively:

$$(U_2)_\lambda E_\lambda + (U_2)_\mu E_\mu + (U_2)_\nu E_\nu = \mu\nu U_\lambda E_\lambda + \nu\lambda U_\mu E_\mu + \lambda\mu U_\nu E_\nu \tag{53}$$

and

$$\begin{aligned}&(\mu+\nu)(U_2)_\lambda E_\lambda + (\nu+\lambda)(U_1)_\mu E_\mu + (\lambda+\mu)(U_2)_\nu E_\nu \\ &\quad = \mu\lambda (U_1)_\lambda E_\lambda + \nu\lambda (U_1)_\mu E_\mu + \lambda\mu (U_1)_\mu E_\mu.\end{aligned} \tag{54}$$

The vectors E_λ, E_μ and E_ν are orthogonal to each other and hence linearly independent. Thus, from (52)–(54), we have the following set of nine equations:

$$\begin{array}{lll} \partial_\lambda(U_1 - (\mu+\nu)U) = 0; & \partial_\lambda(U_2 - \mu\nu U) = 0; & \partial_\lambda((\mu+\nu)U_2 - \mu\nu U_1) = 0 \\ \partial_\mu(U_1 - (\nu+\lambda)U) = 0; & \partial_\mu(U_2 - \lambda\nu U) = 0; & \partial_\mu((\lambda+\nu)U_2 - \lambda\nu U_1) = 0 \\ \partial_\nu(U_1 - (\mu+\lambda)U) = 0; & \partial_\nu(U_2 - \lambda\mu U) = 0; & \partial_\nu((\lambda+\mu)U_2 - \lambda\mu U_1) = 0. \end{array} \tag{55}$$

The general solution of these equations is given by

$$U = \frac{l(\lambda)}{(\lambda-\nu)(\lambda-\mu)} + \frac{m(\mu)}{(\mu-\nu)(\mu-\lambda)} + \frac{n(\nu)}{(\nu-\mu)(\nu-\lambda)} \tag{56}$$

$$U_1 = \frac{(\mu+\nu)l(\lambda)}{(\lambda-\nu)(\lambda-\mu)} + \frac{(\nu+\lambda)m(\mu)}{(\mu-\nu)(\mu-\lambda)} + \frac{(\lambda+\mu)n(\nu)}{(\nu-\mu)(\nu-\lambda)} \tag{57}$$

$$U_2 = \frac{(\mu\nu)l(\lambda)}{(\lambda-\nu)(\lambda-\mu)} + \frac{(\nu\lambda)m(\mu)}{(\mu-\nu)(\mu-\lambda)} + \frac{(\lambda\mu)n(\nu)}{(\nu-\mu)(\nu-\lambda)}, \tag{58}$$

where $l(\lambda)$, $m(\mu)$ and $n(\nu)$ are arbitrary functions of their arguments. From (56)–(58), we obtain the following relations:

$$l(\lambda) = \lambda^2 U - \lambda U_1 + U_2 \tag{59}$$

$$m(\mu) = \mu^2 U - \mu U_1 + U_2 \tag{60}$$

$$n(\nu) = \nu^2 U - \nu U_1 + U_2. \tag{61}$$

Equations (56)–(58) give the solution to the posed problem. In addition, the functions

$$\begin{aligned} h_1 &:= \lambda^2 H - \lambda H_1 + H_2, \\ h_2 &:= \mu^2 H - \mu H_1 + H_2, \\ h_3 &:= \nu^2 H - \nu H_1 + H_2, \end{aligned} \tag{62}$$

commute with each other, a result which is not trivial since λ, μ and ν all depend on $\mathbf{x}$, and that will be used later. Note that equations (59)–(62) lead directly to the separation of the Hamilton–Jacobi equation in the elliptic coordinates λ, μ and ν.

Finally, we would like to mention that some time ago, the case in which the potential U given in (56) satisfies also the Laplace equation, $\triangle U = 0$, was considered in the literature [7].

4. Quantum case

Let us discuss the separation of variables in the quantum case. First, we note that although $g_1^{ik}(\mathbf{x})$ and $g_2^{ik}(\mathbf{x})$ depend on the coordinates, this dependence is quadratic. Matrices $G_i(\mathbf{x}), i = 1, 2$ are symmetric so that the kinetic terms are Hermitian. These are sufficient conditions for the kinetic terms in H_1 and H_2 (see (14) and (15)) are self-adjoint on a proper domain (in fact, they are essentially self-adjoint on the Schwartz space [6]). Thus, the three Hamiltonians, H, H_1 and H_2 are well defined as Hermitian operators in the quantum case.

Then, we are in the position of writing the corresponding Schrödinger equations. For the kinetic terms, it would be convenient to using the following notation:

$$\Delta := -\partial^i \partial_i \tag{63}$$

$$\Delta_1 := p_i g_1^{ik}(\mathbf{x}) p_k = -\partial_i g_1^{ik}(\mathbf{x})\partial_k \tag{64}$$

$$\Delta_2 := p_i g_2^{ik}(\mathbf{x}) p_k = -\partial_i g_2^{ik}(\mathbf{x})\partial_k. \tag{65}$$

Then, as we have made in the classical case, we probe a separation of variables in terms of the ellipsoidal coordinates λ, μ and ν. The common wavefunction of the mutually commuting operators H, H_1 and H_2 will be written as $\psi(\lambda, \mu, \nu)$ in terms of ellipsoidal coordinates. The action of the kinetic operator (63) on $\psi(\lambda, \mu, \nu)$ gives

$$\Delta\psi = \partial_i[\partial_i\lambda\psi_\lambda + \partial_i\mu\psi_\mu + \partial_i\mu\psi_\mu], \tag{66}$$

where ψ_λ, ψ_μ and ψ_ν denote the derivatives of $\psi(\lambda, \mu, \nu)$ with respect to the variables λ, μ and ν, respectively. We now perform the derivative of the term between brackets in (66). Taking into account the orthogonality relations (39) and the definitions given in (26–28), relation (66) yields

$$\Delta\psi = [(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + [(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] + [(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu]. \tag{67}$$

Analogously, the action of (64) on ψ gives

$$\Delta_1\psi = \partial_i g^{ik}[\partial_i\lambda\psi_\lambda + \partial_i\mu\psi_\mu + \partial_i\nu\psi_\nu]. \tag{68}$$

Now, we take into account that the eigenvectors of the matrix $G_1(\mathbf{x})$ (with entries equal $g_1^{ij}(\mathbf{x})$) and their respective eigenvalues are given by (26)–(28).

Next (see (69)), we give two different forms of writing the eigenvalue equations. In each row, we give in the left the eigenvalue equation and in the right the same in coordinate representation. We have

$$\begin{aligned} G_1 E_\lambda &= (\mu+\nu)E_\lambda &\iff\quad& g_1^{ik}\partial_k\lambda = (\mu+\nu)\partial_i\lambda \\ G_1 E_\mu &= (\lambda+\nu)E_\mu &\iff\quad& g_1^{ik}\partial_k\mu = (\lambda+\nu)\partial_i\mu \\ G_1 E_\nu &= (\lambda+\mu)E_\nu &\iff\quad& g_1^{ik}\partial_k\nu = (\lambda+\mu)\partial_i\nu. \end{aligned} \tag{69}$$

Using relations (69) in (68), we get

$$\Delta_1\psi = \partial_i[(\mu+\nu)\partial_i(\lambda\psi_\lambda) + (\lambda+\nu)\partial_i(\mu\psi_\mu) + (\lambda+\mu)\partial_i(\nu\psi_\nu)]. \tag{70}$$

Again, the use of orthogonality relations (39) in (70) gives

$$\begin{aligned} \Delta_1\psi = {}& (\mu+\nu)[(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + (\lambda+\nu)[(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] \\ & + (\lambda+\mu)[(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu]. \end{aligned} \tag{71}$$

Similar manipulations for Δ_2 give

$$\begin{aligned} \Delta_2\psi = {}& \mu\nu[(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + \lambda\nu[(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] \\ & + \lambda\mu[(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu]. \end{aligned} \tag{72}$$

Equations (67), (71) and (72) permit us to write the eigenvalue equations

$$(H-E)\psi = 0; \qquad (H_1-E_1)\psi = 0; \qquad (H_2-E_2)\psi = 0, \tag{73}$$

in the following form:

$$H\psi := [(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + [(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] \\ + [(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu] + 2m[E-U] = 0, \tag{74}$$

$$H_1\psi := (\mu+\nu)[(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + (\lambda+\nu)[(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] \\ + (\lambda+\mu)[(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu] + 2m[E_1-U_1] = 0 \tag{75}$$

and

$$H_2\psi := \mu\nu[(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] + \lambda\nu[(\partial_i\mu)^2\psi_{\mu\mu} + (\Delta\mu)\psi_\mu] \\ + \lambda\mu[(\partial_i\nu)^2\psi_{\nu\nu} + (\Delta\nu)\psi_\nu] + 2m[E_2-U_2] = 0, \tag{76}$$

respectively. At this point, we can use the explicit forms of the potentials U, U_1 and U_2 given in (56)–(58). Then, we replace (56), (57) and (58) in (75), (76) and (77), respectively and observe the relation between the arbitrary function $l(\lambda)$ and the potentials given by (59). Taking this into account, a calculation shows that

$$(\lambda^2 H - \lambda H_1 + H_2)\psi = (\lambda-\mu)(\lambda-\nu)[(\partial_i\lambda)^2\psi_{\lambda\lambda} + (\Delta\lambda)\psi_\lambda] \\ + 2m(\lambda^2 E - \lambda E_1 + E_2 - l(\lambda))\psi = 0. \tag{77}$$

Now, the expression $\Delta = \sum_{i=1}^3 \partial_i^2$ along relations (29)–(31) give[4]

$$\Delta\lambda = 2\frac{a^2b^2 + a^2c^2 + a^2b^2 + 2\lambda(a^2+b^2+c^2) + 3\lambda^2}{(\lambda-\mu)(\lambda-\nu)}. \tag{78}$$

Finally, using (29)–(31) and (77) in (76), we arrive at

$$4(\lambda+a^2)(\lambda+b^2)(\lambda+c^2)\psi_{\lambda\lambda} + 2[a^2b^2 + a^2c^2 + b^2c^2 + 2\lambda(a^2+b^2+c^2) + 3\lambda^2]\psi_\lambda \\ + 2m(\lambda^2 E - \lambda E_1 + E_2 - l(\lambda))\psi = 0. \tag{79}$$

If we factorize $\psi(\lambda,\mu,\nu) = \psi_1(\lambda)\psi_2(\mu)\psi_3(\nu)$, equation (79) gives us $\psi_1(\lambda)$. Similar equations are obtained by just replacing λ by μ and ν. Each equation is an equation that deals with one ellipsoidal variable only, either λ, μ or ν. Thus the procedure of separation of variables in the most general quantum case is completed.

Each of these equations can be written in a more compact form. Let us introduce the following functions that are inverse of each other, $t(\lambda)$ and $\lambda(t)$ with $\lambda(t(\lambda)) = \lambda$. The function $\lambda(t)$ is defined by means of the following condition:

$$\lambda'(t) = 2\sqrt{(\lambda+a^2)(\lambda+b^2)(\lambda+c^2)}, \tag{80}$$

so that

$$t(\lambda) = \int_0^\lambda \frac{dx}{2\sqrt{(x+a^2)(x+b^2)(x+c^2)}}, \tag{81}$$

so that $\lambda(t)$ could be expressed via the Weiersstrass function.

[4] There are similar expressions for $\Delta\mu$ and $\Delta\nu$ which are respectively given by

$$\Delta\mu = 2\frac{a^2b^2 + a^2c^2 + a^2b^2 + 2\mu(a^2+b^2+c^2) + 3\mu^2}{(\mu-\nu)(\mu-\lambda)}$$

and

$$\Delta\nu = 2\frac{a^2b^2 + a^2c^2 + a^2b^2 + 2\nu(a^2+b^2+c^2) + 3\nu^2}{(\nu-\mu)(\nu-\lambda)}.$$

Next, we take a function $f(\lambda(t))$ and let us calculate its second derivative with respect to t. We get

$$\frac{\mathrm{d}f(\lambda(t))}{\mathrm{d}t} = \lambda'(t)\frac{\mathrm{d}f(\lambda(t))}{\mathrm{d}\lambda}, \qquad \frac{\mathrm{d}^2 f(\lambda(t))}{\mathrm{d}t^2} = (\lambda'(t))^2\frac{\mathrm{d}^2 f(\lambda(t))}{\mathrm{d}\lambda^2} + \lambda''(t)\frac{\mathrm{d}f(\lambda(t))}{\mathrm{d}\lambda}$$
$$= 4(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2}$$
$$+ 2[a^2b^2 + a^2c^2 + b^2c^2 + 2\lambda(a^2 + b^2 + c^2) + 3\lambda^2]\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}. \tag{82}$$

We can use the result of (82) in (79) and we finally obtain

$$\frac{\mathrm{d}^2\psi_1(\lambda(t))}{\mathrm{d}t^2} + 2m[\lambda^2(t)E - \lambda(t)E_1 + E_2 - l(\lambda(t))]\psi_1(\lambda(t)) = 0. \tag{83}$$

Then, the problem is solved if we can solve this differential equation. Similar equations give ψ_2 and ψ_3 replacing in (83) λ by μ and ν respectively and $l(\lambda)$ by $m(\mu)$ and $n(\nu)$ respectively.

5. Discussion

We would like to search for solutions for equation (83). First, we fix our attention in equation (80) and make the following change in the unknown function:

$$\Lambda(t) := \lambda(t) - \frac{a^2 + b^2 + c^2}{3}, \tag{84}$$

so that (80) becomes

$$\Lambda'(t) = 2\sqrt{(\Lambda(t) - e_3)(\Lambda(t) - e_2)(\Lambda(t) - e_1)}, \tag{85}$$

where

$$e_1 = \frac{a^2 + b^2 + c^2}{3} - c^2, \qquad e_2 = \frac{a^2 + b^2 + c^2}{3} - b^2, \qquad e_3 = \frac{a^2 + b^2 + c^2}{3} - a^2.$$

Note that $e_1 + e_2 + e_3 = 0$. Then, the general solution of the differential equation (85) is given by [2]

$$\Lambda(t) = \wp(t + \alpha), \tag{86}$$

where $\wp$ is the Weierstrass function [9] and α is an arbitrary integration constant. Thus,

$$\lambda(t) = \wp(t + \alpha) + \frac{a^2 + b^2 + c^2}{3}. \tag{87}$$

At this point, it is convenient to recall that the Weierstrass function is two-periodic with periods $2\omega_1$ and $2\omega_3$ (following the notation in [10]). Then, if we choose $\alpha = \omega_3$, we obtain [10]

$$\wp(t + \omega_3) = -\tfrac{1}{3}(1 + k^2) + k^2\,\mathrm{sn}^2\,t, \tag{88}$$

where

$$k^2 = \tfrac{1}{2}(c^2 + a^2 - 2b^2 + 1) \tag{89}$$

and $\mathrm{sn}\ t$ is the Jacobi elliptic function with modulus k. Then, one choice for $\lambda(t)$ is the following:

$$\lambda(t) = h + k^2\,\mathrm{sn}^2\,t, \qquad \text{with} \qquad h = -\tfrac{1}{2}(a^2 + b^2 + 1). \tag{90}$$

Next, we insert (90) into (83). Then, we have to make a choice for the arbitrary function $l(\lambda(t))$. The point is that (83) is intractable unless that $l(\lambda)$ is quadratic in λ, i.e., $l(\lambda) = \alpha\lambda^2 + \beta\lambda + \gamma$, being α, β and γ constants. In this case, equation (83) becomes

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A + Bk^2\,\mathrm{sn}^2\,t + Ck^4\,\mathrm{sn}^4\,t\right\}\psi_1(\lambda(t)) = 0, \tag{91}$$

where

$$\begin{aligned} A &= 2m[h^2(E-\alpha) - h(E_1+\beta) + \gamma + E_2] \\ B &= 2m[2h(E-\alpha) - -(E_1+\beta)] \\ C &= 2m(E-\alpha). \end{aligned} \tag{92}$$

Equation (91) is known as the ellipsoidal wave equation [11]. Solutions of this equation are given in [11, 12] and are of the form

$$(\mathrm{sn}^p\, t)(\mathrm{cn}^q\, t)(\mathrm{dn}^r\, t)F(\mathrm{sn}^2\, t), \qquad p, q, r = 0, 1, \tag{93}$$

where $\mathrm{sn}\, t$, $\mathrm{cn}\, t$ and $\mathrm{dn}\, t$ are the Jacobi elliptic functions and $F(-)$ is a convergent power series in its argument [11, 12]. There are, however, some simpler solutions. For instance, if we choose $l(\lambda) = E\lambda^2 - E_1\lambda$, we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2mE_2\right]\psi_1(\lambda(t)) = 0 \quad \Longrightarrow \quad \psi_1(\lambda(t)) = A_1\,\mathrm{e}^{\mathrm{i}\sqrt{2mE_2}t} + A_2\,\mathrm{e}^{-\mathrm{i}\sqrt{2mE_2}t}. \tag{94}$$

For $E = \alpha$, we have $C = 0$ and then

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A + Bk^2\,\mathrm{sn}^2\, t\right\}\psi_1(\lambda(t)) = 0. \tag{95}$$

Equation (95) is the Lamé wave equation. Its solutions have been well studied [11, 12]. For the free particle, $l(\lambda) \equiv 0$ and therefore $\alpha = \beta = \gamma = 0$. In this case, the form of equation (83) does not change, as we can see from (92).

6. Concluding remarks

Separation of variables and complete integrability of a three-dimensional system with Hamiltonian given by $H = \mathbf{p}^2/2m + U(\mathbf{x})$ can be achieved in classical mechanics using ellipsoidal coordinates, provided that the potential $U(\mathbf{x})$ belongs to a certain class. The two remainder constants of motion are assumed to have a Hamiltonian structure. We find that their corresponding potentials have to have a similar structure to $U(\mathbf{x})$.

By canonical quantization, we can extend this result to quantum mechanics. However, the three one-dimensional resulting wave equations have the structure of ellipsoidal wave equations, which is one of the most intractable (in the sense of obtaining analytical solutions) types of wave equation, even for the free particle case.

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